THE CONTROL OF CHAOS:
THEORY AND APPLICATIONS

S. BOCCALETTI\textsuperscript{a}, C. GREBOGI\textsuperscript{b}, Y.-C. LAI\textsuperscript{c}, H. MANCINI\textsuperscript{a}, D. MAZA\textsuperscript{a}

\textsuperscript{a}Department of Physics and Applied Mathematics, Institute of Physics, Universidad de Navarra, Irunlarrea s/n, 31080 Pamplona, Spain
\textsuperscript{b}Institute for Plasma Research, Department of Mathematics, and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA
\textsuperscript{c}Dept. of Math. and Electrical Engineering, Center for Systems Science and Engineering Research, Arizona State University, Tempe, AZ 85287
The control of chaos: theory and applications

S. Boccaletti\textsuperscript{a}, C. Grebogi\textsuperscript{b}, Y.-C. Lai\textsuperscript{c}, H. Mancini\textsuperscript{a}, D. Maza\textsuperscript{a}

\textsuperscript{a}Department of Physics and Applied Mathematics, Institute of Physics, Universidad de Navarra, Irunlarrea s/n, 31080 Pamplona, Spain

\textsuperscript{b}Institute for Plasma Research, Department of Mathematics, and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA

\textsuperscript{c}Dept. of Math and Electrical Engineering, Center for Systems Science and Engineering Research, Arizona State University, Tempe, AZ 85287, USA

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Contents

1. Introduction 106
   1.1. The control of chaos: exploiting the critical sensitivity to initial conditions to play with chaotic systems 106
   1.2. From the Ott–Grebogi–Yorke ideas and technique to the other control methods 107
   1.3. Targeting desirable states within chaotic attractors 108
   1.4. The control of chaotic behaviors, and the communication with chaos 109
   1.5. The experimental verifications of chaos control 110
   1.6. Outline of the Report 110
2. The OGY method of controlling chaos 111
   2.1. The basic idea 111
   2.2. A one-dimensional example 111
   2.3. Controlling chaos in two dimensions 114
   2.4. Pole placement method of controlling chaos in high dimensions 121
   2.5. Discussion 127
3. The adaptive method for control of chaos 128
   3.1. The basic idea 128
   3.2. The algorithm for adaptive chaos control 129
   3.3. Application to high-dimensional systems 131
4. The problem of targeting 136
   4.1. Targeting and controlling fractal basin boundaries 136
   4.2. The adaptive targeting of chaos 145
5. Stabilizing desirable chaotic trajectories and application 149
   5.1. Stabilizing desirable chaotic trajectories 149
   5.2. The adaptive synchronization of chaos for secure communication 177
6. Experimental evidences and perspectives of chaos control 179
   6.1. Introduction 179
   6.2. Nonfeedback methods 181
   6.3. Control of chaos with OGY method 182
   6.4. Control of electronic circuits 184
   6.5. Control of chemical chaos 185
   6.6. Control of chaos in lasers and nonlinear optics 186
   6.7. Control of chaos in fluids 187
   6.8. Control of chaos in biological and biomechanical systems 189
   6.9. Experimental control of chaos by time delay feedback 190
   6.10. Other experiments 192
Acknowledgements 192
References 193

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Abstract

Control of chaos refers to a process wherein a tiny perturbation is applied to a chaotic system, in order to realize a desirable (chaotic, periodic, or stationary) behavior. We review the major ideas involved in the control of chaos, and present in detail two methods: the Ott–Greboji–Yorke (OGY) method and the adaptive method. We also discuss a series of relevant issues connected with chaos control, such as the targeting problem, i.e., how to bring a trajectory to a small neighborhood of a desired location in the chaotic attractor in both low and high dimensions, and point out applications for controlling fractal basin boundaries. In short, we describe procedures for stabilizing desired chaotic orbits embedded in a chaotic attractor and discuss the issues of communicating with chaos by controlling symbolic sequences and of synchronizing chaotic systems. Finally, we give a review of relevant experimental applications of these ideas and techniques. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. The control of chaos: exploiting the critical sensitivity to initial conditions to play with chaotic systems

A deterministic system is said to be chaotic whenever its evolution sensitively depends on the initial conditions. This property implies that two trajectories emerging from two different closeby initial conditions separate exponentially in the course of time. The necessary requirements for a deterministic system to be chaotic are that the system must be nonlinear, and be at least three dimensional.

The fact that some dynamical model systems showing the above necessary conditions possess such a critical dependence on the initial conditions was known since the end of the last century. However, only in the last thirty years, experimental observations have pointed out that, in fact, chaotic systems are common in nature. They can be found, for example, in Chemistry (Belousov–Zhabotinski reaction), in Nonlinear Optics (lasers), in Electronics (Chua–Matsumoto circuit), in Fluid Dynamics (Rayleigh–Bénard convection), etc. Many natural phenomena can also be characterized as being chaotic. They can be found in meteorology, solar system, heart and brain of living organisms and so on.

Due to their critical dependence on the initial conditions, and due to the fact that, in general, experimental initial conditions are never known perfectly, these systems are intrinsically unpredictable. Indeed, the prediction trajectory emerging from a bona fide initial condition and the real trajectory emerging from the real initial condition diverge exponentially in course of time, so that the error in the prediction (the distance between prediction and real trajectories) grows exponentially in time, until making the system’s real trajectory completely different from the predicted one at long times.

For many years, this feature made chaos undesirable, and most experimentalists considered such characteristic as something to be strongly avoided. Besides their critical sensitivity to initial conditions, chaotic systems exhibit two other important properties. Firstly, there is an infinite number of unstable periodic orbits embedded in the underlying chaotic set. In other words, the skeleton of a chaotic attractor is a collection of an infinite number of periodic orbits, each one being unstable. Secondly, the dynamics in the chaotic attractor is ergodic, which implies that during its temporal evolution the system ergodically visits small neighborhood of every point in each one of the unstable periodic orbits embedded within the chaotic attractor.

A relevant consequence of these properties is that a chaotic dynamics can be seen as shadowing some periodic behavior at a given time, and erratically jumping from one to another periodic orbit. The idea of controlling chaos is then when a trajectory approaches ergodically a desired periodic orbit embedded in the attractor, one applies small perturbations to stabilize such an orbit. If one switches on the stabilizing perturbations, the trajectory moves to the neighborhood of the desired periodic orbit that can now be stabilized. This fact has suggested the idea that the critical sensitivity of a chaotic system to changes (perturbations) in its initial conditions may be, in fact, very desirable in practical experimental situations. Indeed, if it is true that a small perturbation can give rise to a very large response in the course of time, it is also true that a judicious choice of such a perturbation can direct the trajectory to wherever one wants in the attractor, and to produce a series of desired dynamical states. This is exactly the idea of targeting.
The important point here is that, because of chaos, one is able to produce an infinite number of desired dynamical behaviors (either periodic and not periodic) using the same chaotic system, with the only help of tiny perturbations chosen properly. We stress that this is not the case for a nonchaotic dynamics, wherein the perturbations to be done for producing a desired behavior must, in general, be of the same order of magnitude as the unperturbed evolution of the dynamical variables.

The idea of chaos control was enunciated at the beginning of this decade at the University of Maryland [1]. In Ref. [1], the ideas for controlling chaos were outlined and a method for stabilizing an unstable periodic orbit was suggested, as a proof of principle. The main idea consisted in waiting for a natural passage of the chaotic orbit close to the desired periodic behavior, and then applying a small judiciously chosen perturbation, in order to stabilize such periodic dynamics (which would be, in fact, unstable for the unperturbed system). Through this mechanism, one can use a given laboratory system for producing an infinite number of different periodic behavior (the infinite number of its unstable periodic orbits), with a great flexibility in switching from one to another behavior. Much more, by constructing appropriate goal dynamics, compatible with the chaotic attractor, an operator may apply small perturbations to produce any kind of desired dynamics, even not periodic, with practical application in the coding process of signals.

1.2. From the Ott–Grebogi–Yorke ideas and technique to the other control methods

It is reasonable to assume that one does not have complete knowledge about the system dynamics since our system is typically complicated and has experimental imperfections. It is better, then, to work in the space of solutions since the equations, even if available, are not too useful due to the sensitivity of the dynamics to perturbations. One gets solutions by obtaining a time series of one dynamically relevant variable. The right perturbation, therefore, to be applied to the system is selected after a learning time, wherein the dependence of the dynamics on some external control is tested experimentally. Such perturbation can affect either a control parameter of the system, or a state variable. In the former case, a perturbation on some available control parameter is applied, in the latter case a feedback loop is designed on some state variable of the system.

The first example of the former case is reported in Ref. [1]. Let us draw the attention on a chaotic dynamics developing onto an attractor in a $D$-dimensional phase space. One can construct a section of the dynamics such that it is perpendicular to the chaotic flow (it is called Poincaré section). This $(D-1)$-dimensional section retains all the relevant information of the dynamics, which now is seen as a mapping from the present to the next intersection of the flow with the Poincaré section. Any periodic behavior is seen here as a periodic cycling among a discrete number of points (the number of points determines the periodicity of the periodic orbit). Since all periodic orbits in the unperturbed dynamics are unstable, also the periodic cycling in the map will be unstable. Furthermore, since, by ergodicity, the chaotic flow visits closely all the unstable periodic orbits, this implies that also the mapping in the section will visit closely all possible cycles of points corresponding to a periodic behavior of the system. Let us then consider a given periodic cycle of the map, such as period one. A period one cycle corresponds to a single point in the Poincaré section, which repeats itself indefinitely. Now, because of the instability of the corresponding orbit, this point in fact possesses a stable manifold and an unstable manifold. For stable (unstable) manifold we mean the collection of directions in phase space through which the trajectory
approaches (diverges away from) the point geometrically. The control of chaos idea consists in perturbing a control parameter when the natural trajectory is in a small neighborhood of the desired point, such that the next intersection with the Poincaré section puts the trajectory on the stable manifold. In this case, all divergences are cured, and the successive natural evolution of the dynamics, except for nonlinearities and noise, converges to the desired point (that is, it stabilizes the desired periodic behavior). Selection of the perturbation is done by means of a reconstruction from experimental data of the local linear properties of the dynamics around the desired point.

In some practical situations, however, it may be desirable to perform perturbations on a state variable accessible to the operator. This suggests the development of some alternative approaches. The first was introduced in Ref. [2]. It consists in designing a proper feedback line through which a state variable is directly perturbed such as to control a periodic orbit. This second method requires the availability of a state variable for experimental observation and for the perturbations. In such a case, a negative feedback line can be designed which is proportional to the difference between the actual value of the state variable, and the value delayed of a time lag $T$. The idea is that, when $T$ coincides with the period of one unstable periodic orbit of the unperturbed system, the negative feedback pushes to zero the difference between the present and the delayed dynamics, and the periodic orbit is stabilized. Furthermore, as soon as the control becomes effective, this difference goes effectively to zero, so that the feedback perturbation vanishes. Moreover, as before, a preliminary learning time is needed, for learning the periods of the unstable periodic orbits. In the above mechanism, the proportionality constant entering in the feedback loop is given in Ref. [3] where an adaptive technique has been introduced which automatically selects this constant by adaptively exploiting the local dynamics of the system.

Many other techniques have been introduced with the aim of establishing control over chaos that will be referred to and described along this Report. Among the many available reviews, books, and monographies on this matter, here we address the reader the most recent ones, contained in Refs. [4–8]. In face of this huge number of theoretical studies, experimental realizations of chaos control have been achieved with a magnetoelastic ribbon [9], a heart [10,11], a thermal convection loop [12,13], a yttrium iron garnet oscillator [14], a diode oscillator [15], an optical multimode chaotic solid-state laser [16], a Belousov–Zhabotinski reaction diffusion chemical system [17], and many other experiments.

While control of chaos has been successfully demonstrated experimentally in many situations, the control of patterns in space-extended systems is still an open question. This is the reason why most of the interest has moved actually from the control of periodic behaviors in concentrated systems, to the control of periodic patterns in space-extended systems, with the aim of controlling infinite dimensional chaos, or even space–time chaos. The applications would be enormous, ranging from the control of turbulent flows, to the parallel signal transmission and computation to the parallel coding-decoding procedure, to the control of cardiac fibrillation, and so forth.

1.3. Targeting desirable states within chaotic attractors

One of the major problems in the above process is that one can switch on the control only when the system is sufficiently close to the desired behavior. This is warranted by the ergodicity of chaos regardless of the initial condition chosen for the chaotic evolution, but it may happen that the small neighborhood of a given attractor point (target) may be visited only infrequently, because